A Generalized Duality and Applications

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Abstract. The aim of this paper is to present a nonconvex duality with a zero gap and its connection with convex duality. Since a convex program can be regarded as a particular case of convex maximization over a convex set, a nonconvex duality can be regarded as a generalization of convex duality. The generalized duality can be obtained on the basis of convex duality and minimax theorems. The duality with a zero gap can be extended to a more general nonconvex problems such as a quasiconvex maximization over a general nonconvex set or a general minimization over the complement of a convex set. Several applications are given.

Key words. Nonconvex duality, zero gap, global optimization.

1. Introduction

Over the last decades duality theory has been fully developed for general convex minimization problems. The duality is constructed on the basis of the Karush–Kuhn–Tucker condition and Lagrangian multipliers. For a given convex program we can define a variety of dual problems depending on the way of perturbation. One of the most often used perturbations is the Lagrangian relaxation. For more general programs this approach can be generalized by using a concept of price functions (see, e.g. [24, 2]). In principle we could obtain a dual problem of a nonconvex optimization problem by using the class of all nondecreasing price functions. However, the dual is no longer defined in a finite dimensional space. Many attempts have been made to provide a suitable class of price functions whose parameters are in a finite dimensional space, but the duality gap cannot be then avoided in general nonconvex problems. Although there are many significant results on an estimation of the size of the duality gap (see, e.g. [1, 6]), this is an obstacle for algorithmic implementations.

In Global Optimization, where local optima are not global, the algorithmic studies have typical been developed based on the combinatorial techniques such as enumeration and branch and bound methods. We mention the pioneering paper by Tuy [26] and the excellent survey by Horst and Tuy [7] in this field. Convex duality can be incorporated into the solution methods for nonconvex problems for obtaining the valid cuts or the lower bounds. However, the conceptual aspect of dualization in nonconvex problems seems to have a big difference with respect to convex duality. The first nonconvex-type duality was

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introduced by Toland [25]. This dualization was used earlier by Pshenichnyyi [14] in differential game and optimal control, and was developed later to further applications of d.c. minimization (abbreviation of "minimization of the difference of two convex functions on a convex set") by Hiriart-Urruty [5]. More recently, another nonconvex-type duality has been introduced [18]. There are several points of differences between Toland's duality and our duality. In our duality one can obtain a dual problem of a reverse convex program (abbreviation of "convex minimization over the complement of a convex set") which is more general than d.c. minimization.

In this paper we shall see a connection between nonconvex duality and convex duality. If we consider a concave program (abbreviation of "convex maximization over a convex set") as a generalization of convex programs then we can obtain its dual problem by using convex duality and the minimax theorem. Here we would mention that the term "concave program" is used in many literatures (see [7] and its references) and it comes quite reasonably because *convex maximization* is equivalent to *concave minimization*. The transformations for a dual problem of a concave program presents a natural route to a definition of the so-called quasiconjugate. In this paper the duality will be extended to more general problems such as a quasiconvex maximization over a compact set or a general minimization over the complement of a convex set. Several applications are given.

The paper is organized as follows. In Section 2 we give a connection between convex duality and a nonconvex duality. We present a way to define a dual problem of a concave program by using convex duality and the minimax theorem. In Section 3 we present a duality for generalized concave and reverse convex programs, where the objectives are quasiconvex. Several applications are given in Section 4. In Section 5 we present extensions and finally we draw some conclusions in Section 6.

2. A Dual Problem of a Concave Program

Let us consider a convex maximization over a convex set

$$\sup\{f(x) : x \in X\} \tag{1}$$

where f is a convex function and X a compact convex set containing $0 \in R^n$. Since convex maximization is equivalent to concave minimization, problem (1) is often called a concave program (see [7]). Since a linear function is, of course, a convex function, the convex program

$$\sup\{\langle c, x\rangle : x \in X\}\ , \tag{2}$$

with $c \in R^n$ is a particular case of program (1). We have

$$\sup\{\langle c, x\rangle : x \in X\}$$

$$= \inf\{t : t \geq \langle c, x\rangle \forall x \in X\rangle$$

$$= 1/\sup\{t : 1/t \geq \langle c, x\rangle \forall x \in X\}$$
$$= 1/\sup\{t : 1 \geq \langle tc, x\rangle \forall x \in X\}$$
$$= 1/\sup\{t : tc \in X^0\}\,, \tag{3}$$

where X^0 denotes the polar of X:

$$X^0 = \{v : \langle v, x\rangle \leq 1\ \forall x \in X\}\,.$$

Since X is compact, $0 \in \operatorname{int} X^0$. We call now the problem

$$\sup\{\lambda : \lambda c \in X^0\} \tag{4}$$

a dual problem of problem (2). The vector c in the primal problem (2) is regarded as a linear function defined on the variable space, whereas it is an element in the variable space of the dual problem (4). Therefore it is reasonable to use here the term "dual". If the feasible domain X is given by a system of convex (or, simpler, linear) inequations, then it has been known that by using representations of the polar set X^0 we can obtain more sophisticated dual problems. We consider now other equivalent formulations of the dual problem (4). Since X^0 is a closed convex set containing 0 in its interior, problem (4) is equivalent to

$$\inf\{\lambda > 0 : \lambda c \not\in \operatorname{int} X^0\}\,. \tag{5}$$

It is well known that there are two ways of representation of a given closed convex set V. The convex hull of points in V is a direct representation, and the intersection of closed halfspaces containing V is a dual representation. The equivalence between (4) and (5) is based on the direct representation of the convex set $V = X^0$. Since $0 \in \operatorname{int} V$, one has $\lambda v \in \operatorname{int} V$ for all $v \in V$ and $\lambda \in [0, 1)$. The proof of this property is based on the direct representation of the convex set V ([15], Theorem 6.1). This property implies that

$$\sup\{t : tc \in V\} \leq \inf\{t > 0 : tc \not\in \operatorname{int} V\}\,.$$

The inverse inequality is simply obtained from the fact that for every $t \geq 0$, one has either $tc \in V$ or $tc \not\in \operatorname{int} V$.

Now we define

$$\varphi_c(v) = \inf\{\langle c, x\rangle : \langle v, x\rangle \geq 1\} \quad \forall v \in R^n\,. \tag{6}$$

It is easy to check that $\varphi_c(v)$ is quasiconcave w.r.t. v and

$$\varphi_c(v) = \begin{cases} 1/\lambda & \text{if } v = \lambda c\,, \quad \lambda \geq 0 \\ -\infty & \text{otherwise}\,. \end{cases}$$

Therefore,

$$\sup\{\langle c, x\rangle : x \in X\}$$
$$= 1/\inf\{\lambda > 0 : \lambda c \not\in \operatorname{int} X^0\}$$
$$= \sup\{1/\lambda : \lambda c \not\in \operatorname{int} X^0\}$$

$$= \sup_{(\lambda, v)} \{1/\lambda : \lambda c = v,\ v \not\in \operatorname{int} X^0\}$$

$$= \sup_{v \not\in \operatorname{int} X^0} \varphi_c(v)\,.$$

The problem

$$\sup_{v \not\in \operatorname{int} X^0} \varphi_c(v) \tag{7}$$

is a maximization of a quasiconcave function (or, equivalently, a minimization of a quasiconvex function) over the complement of a convex set and it is called a dual problem of (2). The dual problems (4), (5) and (7) are equivalent and the simplest one (4) is more often used in convex duality. However, formulation (7) will be used in the sequel to obtain a dual problem of a general concave program.

We reconsider a concave program (1) where the convex function $f(\cdot)$ is given by the supremum of a family of affine functions:

$$f(x) = \sup\{\langle v, x\rangle - g(v) : v \in G\} \tag{8}$$

where G is a closed convex set and $g(\cdot)$ is a closed convex function w.r.t. v. If f is a proper closed convex function, then $f = f^{**}$ and it can be represented as in (8) where $g(v) = f^*(v)$ and $G = \operatorname{dom} g^*$ (see, e.g. [15]). We have

$$\sup\{f(x) : x \in X\}$$

$$= \sup_{x \in X} \sup_{v \in G} \{\langle v, x\rangle - g(v)\}$$

$$= \sup_{v \in G} \sup_{x \in X} \{\langle v, x\rangle - g(v)\}$$

$$= \sup_{v \in G} \{-g(v) + \sup_{x \in X} \langle v, x\rangle\}$$

$$= \sup_{v \in G} \{-g(v) + \sup_{z \not\in \operatorname{int} X^0} \varphi_v(z)\} \quad (*)$$

$$= \sup_{v \in G} \{-g(v) + \sup_{z \not\in \operatorname{int} X^0} \inf\{\langle v, x\rangle : \langle z, x\rangle \geq 1\}\}$$

$$= \sup_{v \in G} \sup_{z \not\in \operatorname{int} X^0} \{-g(v) + \inf_x \{\langle y, x\rangle : \langle z, x\rangle \geq 1\}\}$$

$$= \sup_{z \not\in \operatorname{int} X^0} \sup_{v \in G} \inf_{x : \langle z, x\rangle \geq 1} \{\langle v, x\rangle - g(v)\}$$

$$= \sup_{z \not\in \operatorname{int} X^0} \inf_{x : \langle z, x\rangle \geq 1} \sup_{v \in G} \{\langle v, x\rangle - g(v)\} \quad (**)$$

$$= \sup_{z \not\in \operatorname{int} X^0} \inf_x \{f(x) : \langle z, x\rangle \geq 1\}\,.$$

In the above transformations we use a convex duality on the line (*) and the minimax theorem on the line (**). If we denote

$$\varphi(z) = \inf_z \{f(x) : \langle z, x\rangle \geq 1\}\,, \tag{9}$$

then $\varphi(\cdot)$ is quasiconcave w.r.t. z and

$$\sup\{f(x) : x \in X\}$$
$$= \sup\{\varphi(z) : z \not\in \operatorname{int} X^0\} . \tag{10}$$

Problem (10) is called a dual problem of a general concave program (1) and it is a maximization of a quasiconcave function (or, equivalently, a minimization of a quasiconvex function) over the complement of a convex set. The objective function $\varphi(\cdot)$ defined via (9) is related to a quasiconjugation [18].

3. A Generalized Duality

In this section we present a generalized duality for a quasiconvex maximization over a compact convex set (briefly called a generalized concave program), and a quasiconvex minimization over the complement of a convex set (briefly called a generalized reverse convex program). There are several reasons for us to consider quasiconvex functions which are more general than convex functions. The first reason is that the class of quasiconvex minimization on the complement of a convex set includes the quasiconvex minimization over a convex set, which is a multi-extremal optimization problem, but reasonably is the first generalization of convex minimization. The second reason is that the level sets play a more crucial role than the epigraphs in nonconvex problems and the class of convex functions is not large enough and not stable w.r.t. the quasiconjugation, which is an important instrument in a nonconvex duality.

Let us recall here a definition of the quasiconjugate [18].

DEFINITION 3.1. Let $f : R^n \to R \cup \{\pm\infty\}$. The function $f^H : R^n \to R \cup \{\pm\infty\}$ defined by

$$f^H(v) = \begin{cases} -\inf\{f(x) : \langle v, x\rangle \geq 1\} & \text{if } v \neq 0 \\ -\sup\{f(x) : x \in R^n\} & \text{if } v = 0 \end{cases}$$

is called the quasiconjugate of f. The function $(f^H)^H$ is called the biquasiconjugate of f and briefly denoted by f^{HH}.

An important class is of functions f such that $f = f^{HH}$. Denote by Φ this class. Since f^{HH} is a quasiconvex function satisfying

$$f^{HH}(0) = \inf\{f^{HH}(x) : x \in R^n \backslash \{0\}\} ,$$

if $f \in \Phi$ then f is a quasiconvex function satisfying

$$f(0) = \inf\{f(x) : x \in R^n \backslash \{0\}\} . \tag{11}$$

Denote by Ψ the class of quasiconvex functions $f : R^n \to R \cup \{\pm\infty\}$ satisfying (11). Ψ obviously contains Φ. However, the difference between Ψ and Φ is not very big. Function $f \in \Psi$ will belong to Φ if f is either usc (see [18], Theorem 4.1) or lsc (see Theorem 3.1 below).

THEOREM 3.1. *If $f \in \Psi$ is lsc, then $f \in \Phi$, i.e. $f^{HH} = f$.*

Proof. Since $f \in \Psi$, one has

$$f(0) = f^{HH}(0)$$
$$f(x) \geq f^{HH}(x) \quad \forall x \in R^n$$

([18], Lemma 4.1). Suppose that $f^{HH}(x) < f(x)$ at some $x \neq 0$. There is a vector v such that

$$1 = \langle v, x \rangle > \sup\{\langle v, y \rangle : f(y) \leq f^{HH}(x)\} .$$

Such a vector v exists, because f is lsc, $f(x) > f^{HH}(x)$ and $f(0) \leq f^{HH}(x)$. So,

$$\inf\{f(y) : \langle v, y \rangle \geq 1\} > f^{HH}(x)$$
$$\Rightarrow f^H(v) = -\inf\{f(y) : \langle v, y \rangle \geq 1\} < -f^{HH}(x) . \tag{12}$$

On the other hand one has

$$f^{HH}(x) = -\inf\{f^H(u) : \langle u, x \rangle \geq 1\} \geq -f^H(v) ,$$

because $\langle v, x \rangle = 1$. This together with (12) implies $f^{HH}(x) > f^{HH}(x)$. This is a contradiction. Thus, $f^{HH} = f$. □

A problem of maximizing a function $f \in \Phi$ over a compact convex set X containing 0:

$$\max\{f(x) : x \in X\} \tag{13}$$

is called a generalized concave program, and a problem of minimizing a function $f \in \Phi$ over the complement of an open convex set Y containing 0:

$$\min\{f(x) : x \not\in Y\} \tag{14}$$

is called a generalized reverse convex program. We would mention here that a concave program can be simply transformed into a reverse convex program by using an additional variable (see, e.g. [7]). However, a generalized concave program cannot be simply transformed into a generalized reverse convex program. A dual problem of (13), by definition, is

$$\min\{f^H(v) : v \not\in \operatorname{int} X^0\} \tag{15}$$

and a dual problem of (14), by definition, is

$$\max\{f^H(v) : v \in Y^0\} . \tag{16}$$

Since $(f^H)^{HH} = (f^{HH})^H = f^H$, i.e., $f^H \in \Phi$ if $f \in \Phi$, the dual (15) is a generalized reverse convex program, and the dual (16) is a generalized concave program. If we take now a dual of problem (15), then we have

$$\max\{f^{HH}(x) : x \in (\operatorname{int} X^0)^0\}$$
$$\Leftrightarrow \max\{f(x) : x \in X\} .$$

Therefore the dual of the dual is exactly the primal.

THEOREM 3.2. *If $f \in \Phi$ and X is a compact convex set containing* 0 *then*

$$\sup\{f(x) : x \in X\} = -\inf\{f^H(v) : v \not\subseteq \operatorname{int} X^0\} .$$

Proof. We have

$$\begin{aligned}
&\sup\{f(x) : x \in X\} \\
&= \sup\{f^{HH}(x) : x \in X\} \\
&= \sup_x \{-\inf_v \{f^H(v) : \langle v, x\rangle \geq 1\} : x \in X\} \\
&= -\inf_{x \in X} \inf_v \{f^H(v) : \langle v, x\rangle \geq 1\} \\
&= -\inf_v \inf_{x \in X} \{f^H(v) : \langle v, x\rangle \geq 1\} \\
&= -\inf_v \{f^H(v) : v \not\subseteq \operatorname{int} X^0\}
\end{aligned}$$

(because $v \not\subseteq \operatorname{int} X^0 \Leftrightarrow \exists x \in X : \langle v, x\rangle \geq 1$) . □

We now discuss the solvability of the generalized concave programs, the generalized reverse convex programs, and the relationships between optimal solutions of the primal and dual problems. By the symmetricity we can consider only the primal-dual pair (13), (15). In problem (13) if the function f is usc, then there is $x^* \in X$ such that $f(x^*) = \max\{f(x) : x \in X\}$, and in problem (15) if the function f^H is lsc and satisfies the following coercivity condition:

$$f^H(v) \to \sup\{f^H(u) : u \in R^n\} \text{ as } \|v\| \to \infty , \tag{17}$$

then there is $v \not\subseteq \operatorname{int} X^0$ such that

$$f^H(v) = \min\{f^H(u) : u \not\subseteq \operatorname{int} X^0\} .$$

Furthermore, f is usc if and only if f^H is lsc and satisfying (17) (see [18]). We now have the following duality theorem.

THEOREM 3.3. *Assume that $f \in \Phi$ is usc.*

(*i*) *The primal problem* (13) *and its dual problem* (15) *are solvable.*

(*ii*) *If v^* solves the dual* (15) *then any vector in the convex set $\{x \in X : \langle v^*, x\rangle \geq 1\}$ solves the primal* (13).

(*iii*) *If x^* solves the primal* (13) *then every minimizer of f^H on the halfspace $\{v : \langle x^*, v\rangle \geq 1\}$ solves the dual* (15).

Proof. The proof of assertions (i) and (ii) can be found in [18]. It remains to prove assertion (iii). Suppose that x^* solves (13) and v^* is a minimizer of f^H on the halfspace $\{v : \langle x^*, v\rangle \geq 1\}$ (such a vector exists by [18], Lemma 3.1). Then,

$$\begin{aligned} f^H(v^*) &= \inf\{f^H(v) : \langle x^*, v\rangle \geq 1\} \\ &= -f^{HH}(x^*) \\ &= -f(x^*) \\ &= -\sup(13) \\ &= -\inf(15)\,. \end{aligned}$$

Further v^* is feasible to (15), because $x \in X$ and $\langle x^*, v^*\rangle \geq 1$. Therefore v^* solves (15). □

EXAMPLE 3.1. $f(x) = \max\{\langle c, x\rangle, 0\}$. Then $f \in \Phi$ and

$$f^H(v) = \begin{cases} -\infty & \text{if } v = 0 \\ -\dfrac{1}{\lambda} & \text{if } v = \lambda c \text{ for some } \lambda > 0 \\ +\infty & \text{otherwise}\,. \end{cases}$$

Primal:

$$\max\{\langle c, x\rangle : x \in X\} \Leftrightarrow \max\{f(x) : x \in X\}\,.$$

Dual:

$$\min\{f^H(v) : v \not\in \operatorname{int} X^0\} \Leftrightarrow \min\{\lambda > 0 : \lambda c \not\in \operatorname{int} X^0\}\,.$$

The dual is exactly problem (5) which could be obtained by convex duality.

4. Applications

APPLICATION 4.1. We consider the following problems.

Problem (A). Let X be a bounded polyhedral convex set given by

$$X = \{x \in R^n : \langle a_i, x\rangle \leq 1\ \forall i = 1, \ldots, k\}\,, \tag{18}$$

where $a_i \in R^n$ $i = 1, \ldots, k$. Find the smallest ball centered at 0 containing X.

Problem (B). Let V be a polyhedral convex set given by

$$V = \operatorname{conv}\{a_1, \ldots, a_k\}\,, \tag{19}$$

where $a_i \in R^n$ $i = 1, \ldots, k$. Find the biggest ball centered at 0 contained in V. Problem (A) can be formulated as follows

$$\sup\{\|x\|^2 : \langle a_i, x\rangle \leq 1\ \forall i = 1, \ldots, k\}\,. \tag{20}$$

This is a linearly constrained quadratic concave program. The quasiconjugate of the function $x \mapsto \|x\|^2$ is $v \mapsto -1/\|v\|^2$, and the polar of X is V. So the dual of (20) is

$$\inf\left\{-\frac{1}{\|y\|^2} : y \not\in \operatorname{int} V\right\}$$

$$\Leftrightarrow \inf\{\|y\|^2 : y \not\in \operatorname{int} V\}\,.$$

This is problem (B). If x^* is a solution of problem (A), then by Theorem 3.3, $x^*/\|x^*\|^2$ is a solution of problem (B).

APPLICATION 4.2. We consider a bilinear programming

$$\sup_{g(x)\leq 1} \sup_{h(z)\leq 1} z^T Ax \tag{21}$$

where A is a nonsingular $n \times n$ matrix, $g(x)$ and $h(z)$ are the optimal values of the following convex programs

$$g(x) = \sup\{u^T Mx : u \in U\}$$
$$h(z) = \sup\{z^T Nv : v \in V\}$$

with U, V being compact convex sets containing 0 in R^n, and M, N matrices of the sizes $n \times n$. Set

$$f(x) = \sup\{z^T Ax : h(z) \leq 1\}$$
$$X = \{x : g(x) \leq 1\} .$$

Problem (21) is then to maximize f over X. The polar V of X, and the quasiconjugate f^H can be defined as follows

$$V = \{M^T u : u \in U\}$$
$$-f^H(v) = \inf_x \sup_z \{z^T Ax : h(z) \leq 1, \langle x, v\rangle \geq 1\}$$
$$= \sup_z \inf_x \{z^T Ax : h(z) \leq 1, \langle x,v\rangle \geq 1\} .$$

For any z such that $A^T z \not\in \{\lambda v : \lambda \geq 0\}$ we have

$$\inf\{z^T Ax : \langle x, v\rangle \geq 1\} = -\infty .$$

For $z = \lambda(A^T)^{-1}v (\lambda \geq 0)$ we have

$$\inf\{z^T Ax : \langle x, v\rangle \geq 1\} = \lambda .$$

Therefore,

$$-f^H(v) = \sup\{\lambda : h(\lambda(A^T)^{-1}v) \leq 1\}$$
$$= \sup\{\lambda : \lambda h((A^T)^{-1}v) \leq 1\}$$
$$= \frac{1}{h((A^T)^{-1}v)}$$

So, the dual is

$$\inf\{-\frac{1}{h((A^T)^{-1}v)} : v \not\in \text{int } Y\}$$

$$\Leftrightarrow \inf\{h((A^T)^{-1}v) : v \not\in \text{int } Y\}$$

$$\Leftrightarrow \inf\{h((A^T)^{-1}M^T u) : u \not\in \text{int } U\}$$

$$\Leftrightarrow \inf_{u \not\in \text{int } U} \sup_{v \in V} u^T MA^{-1}Nv .$$

We obtain a duality relationship between a bilinear programming and a minimax problem with a reverse convex constraint.

APPLICATION 4.3. Consider the following program

$$\min\{f_0(x) : f_1(x) \times \cdots \times f_k(x) \leq 1, x \in A\}, \tag{22}$$

where A is a compact convex set in R^n, f_0 is a convex function, f_i, $i = 1, \ldots, k$ are nonnegative-valued convex functions on A and k an integer smaller than n. Let w be a minimizer of the convex function f_0 on the convex set A. If $f_1(w) \times \cdots \times f_k(w) \leq 1$, then w solves (22). This case is not of our interests. We are interested in the following case

$$f_1(w) \times \cdots \times f_k(w) > 1. \tag{23}$$

Program (22) is equivalent to the following reverse convex program

$$\min\{f_0(x) : t_1 \times \cdots \times t_k \leq 1, f_i(x) \leq t_i\ i = 1, \ldots, k, x \in A\}. \tag{24}$$

In program (24) the reverse convex constraint is

$$t_1 \times \cdots \times t_k \leq 1.$$

By translating $0 \in R^n \times R^k$ to $(0, f_1(w), \ldots, f_k(w)) \in R^n \times R^k$, program (24) can be transformed into

$$\min\{f_0(x) : \prod_{i=1}^{k} (t_i + f_i(w)) \leq 1, f_i(x) - f_i(w) \leq t_i\ i = 1, \ldots, k, x \in A\}. \tag{25}$$

Set

$$D = \{(x, t) \in R^n \times R^k : f_i(x) - f_i(w) \leq t_i\ = 1, \ldots, k\ x \in A\}$$

$$Y = \{(x, t) : \prod_{i=1}^{k} (t_i + f_i(w)) \geq 1, t_i + f_i(w) \geq 0\ i = 1, \ldots, k\}$$

$$g(x, t) = f_0(x) + \delta((x, t)|D),$$

where $\delta(\cdot|D)$ is the indicator of D. Program (25) can be then rewritten as follows

$$\inf\{g(x, t) : (x, t) \not\in \operatorname{int} Y\}.$$

Since function $g(x, t)$ is convex, this is a reverse convex program. It can be checked that

$$Y^0 = \{(0, \lambda) \in R^n \times R^k : -kx \left(\prod_1^k (-\lambda_i)\right)^{1/k} - \lambda_i f_i(w) \leq 1,$$

$$\lambda = (\lambda_1, \ldots, \lambda_k) \leq 0\} \subseteq 0 \times R^k_- \tag{26}$$

and the quasiconjugate of g is defined on $0 \times R^k$ as follows

$$\begin{aligned} g^H(\lambda) &= -\inf\{g(x,t) : \langle t, \lambda\rangle \geqslant 1\} \\ &= -\inf\{f_0(x) + \delta((x,t)|D) : \langle t,\lambda\rangle \geqslant 1\} \\ &= -\inf\{f_0(x) : \langle t,\lambda\rangle \geqslant 1, (x,t)\in D\} \\ &= -\inf\{f_0(x) : \sup_{x\in A} \sum_{i=1}^{k} \lambda_i(f_i(x) - f_i(w)) \geqslant 1\} \end{aligned}$$

Therefore the dual of (25) is

$$\sup\{g^H(\lambda) : \lambda \in Y^0\}\,.$$

This is a generalized concave program in R^k. When $k = 2$ it can be solved by a practically efficient algorithm [19].

5. Extensions

The generalized duality can be applied to more general nonconvex optimization problems.

Using the duality presented in the previous section we can obtain a dual problem of a quasiconvex maximization problem over a compact (nonconvex) set

$$\sup\{f(x) : x \in X\}\,, \tag{27}$$

where $f \in \Phi$ and X is a compact set such that $0 \in \operatorname{conv} X$. Since f is quasiconvex, we have

$$\sup\{f(x) : x \in X\} = \sup\{f(x) : x \in \operatorname{conv} X\}\,.$$

Since $X^0 = (\operatorname{conv} X)^0$, the dual of problem $\sup\{f(x) : x \in \operatorname{conv} X\}$ is

$$\inf\{f^H(v) : v \not\in \operatorname{int} X^0\}\,. \tag{28}$$

If v^* solves (28), then the set $\{x \in \operatorname{conv} X : \langle v^*, x\rangle \geqslant 1\}$ is nonempty and every vector in this set maximizes $f(\cdot)$ on $\operatorname{conv} X$. This implies that every vector $x \in X$ such that $\langle v^*, x\rangle \geqslant 1$ maximizes $f(\cdot)$ on X. The problem (28) is called a dual of problem (27).

We consider now a general minimization over the complement of a convex set

$$\inf\{f(x) : x \not\in \operatorname{int} X\}\,, \tag{29}$$

where X is a closed convex set satisfying $0 \in \operatorname{int} X$, and f is a function satisfying $f(0) = \inf\{f(x) : x \in R^n \setminus \{0\}\}$.

THEOREM 5.1. *If either f is usc or f is lsc and satisfies $f(x) \to \sup\{f(x) : x \in R^n\}$ as $\|x\| \to \infty$, then*

$$\inf\{f(x) : x \not\in Y\} = \inf\{f^{HH}(x) : x \not\in Y\} \tag{30}$$

for every open convex set Y.

Proof. Suppose that f is usc. Then,

$$\{x : f^{HH}(x) < \alpha\} = \text{conv}\{x : f(x) < \alpha\} \quad \forall\alpha$$

(see [18]). Therefore,

$$\begin{aligned}
&\alpha \leqslant \inf\{f(x) : x \not\in Y\} \\
&\Leftrightarrow \{x : f(x) < \alpha\} \subseteq Y \\
&\Leftrightarrow \text{conv}\{x : f(x) < \alpha\} \subseteq Y \\
&\Leftrightarrow \{x : f^{HH}(x) < \alpha\} \subseteq Y \\
&\Leftrightarrow \leqslant \{f^{HH}(x) : x \not\subseteq Y\} .
\end{aligned}$$

So we have (30). If f is lsc and $f(x) \rightarrow \sup\{f(x) : x \in R^n\}$ as $\|x\| \rightarrow \infty$, then f has a minimizer on every closed set, and

$$\{x : f^{HH}(x) \leqslant \alpha\} = \text{conv}\{x : f(x) \leqslant \alpha\}$$

[18]. Therefore

$$\begin{aligned}
&\alpha < \min\{f(x) : x \not\subseteq Y\} \\
&\Leftrightarrow \{x : f(x) \leqslant \alpha\} \subseteq Y \\
&\Leftrightarrow \text{conv}\{x : f(x) \leqslant \alpha\} \subseteq Y \\
&\Leftrightarrow \{x : f^{HH}(x) \leqslant \alpha\} \subseteq Y \\
&\Leftrightarrow \alpha < \min\{f(x) : x \not\subseteq Y\} .
\end{aligned}$$

Then we also have (30). □

With the assumption of Theorem 5.1, we have

$$\inf\{f(x) : x \not\subseteq \text{int } X\} = \inf\{f^{HH}(x) : x \not\subseteq \text{int } X\} .$$

Since $f^{HHH} = f^H$, the dual of the problem $\inf\{f^{HH}(x) : x \not\subseteq \text{int } X\}$ is

$$\sup\{f^H(v) : v \in X^0\} . \tag{31}$$

If v^* solves (31), then every minimizer of $f^{HH}(\cdot)$ on the halfspace $\{x : \langle v^*, x\rangle \geqslant 1\}$ is also a minimizer of $f^{HH}(\cdot)$ on $R^n \backslash \text{int } X$. Therefore, every minimizer of $f(\cdot)$ on $\{x : \langle v^*, x\rangle \geqslant 1\}$ is also a minimizer of $f(\cdot)$ on $R^n \backslash \text{int } X$. The problem (31) is also called a dual problem of problem (29).

6. Discussions

In this paper we have shown that if we consider a convex program as a particular case of a concave program, then the dual problems obtained by convex duality and a nonconvex duality are equivalent. Further, a dual problem of a concave program can be obtained on the basis of convex duality and the minimax theorem. Therefore, in some senses the nonconvex duality can be regarded as a generalization of convex duality.

For another class of nonconvex optimization

$$\inf_{x \in R^n} \{h_1(x) - h_2(x)\} \tag{32}$$

where h_1 is an arbitrary function and h_2 is a finite convex function, Toland [25] introduced a dual problem

$$\inf_{y \in \mathrm{dom}\, h_2^*} \{h_2^*(y) - h_1^*(y)\}\,. \tag{33}$$

By using an additional variable t we can transform problem (32) into a minimization on the complement of an open convex set in R^{n+1}. In [21] we show that the dual problem (33) can be obtained from our duality.

The duality theory plays an important role in constructing efficient computational methods. In many cases the dual problem is much simpler than the primal problem. For instance, the number of variables in the dual problem is much smaller than the number of variables in the primal problem. The approach using this duality has recently been used to solve certain classes of large-scale nonconvex optimization (see [19, 20, 22]).

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